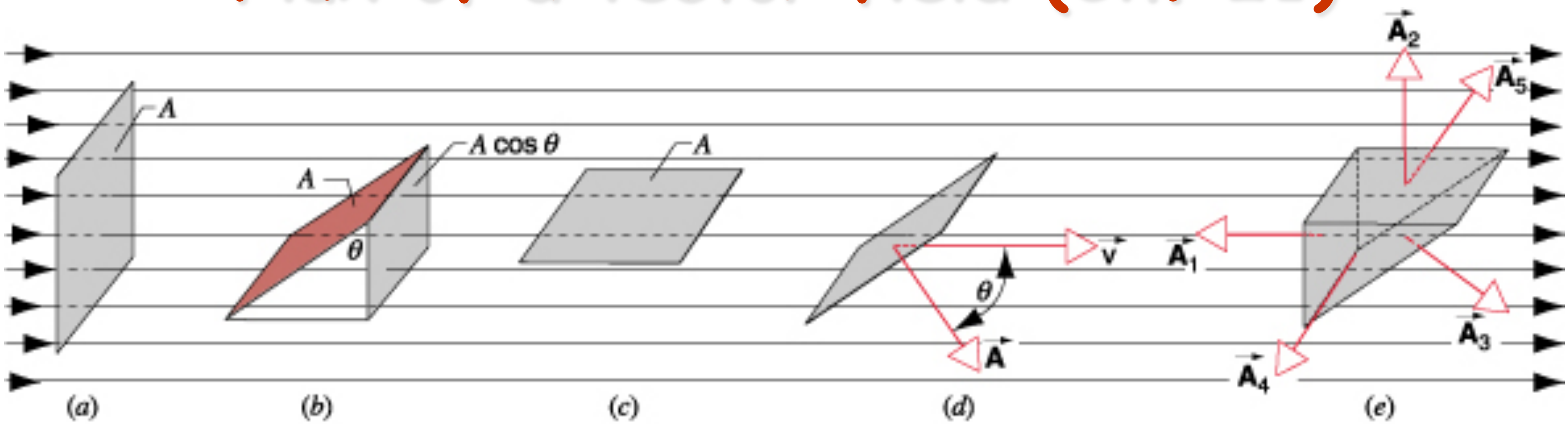
Chapter 21: Gauss' law
Thursday September 8th

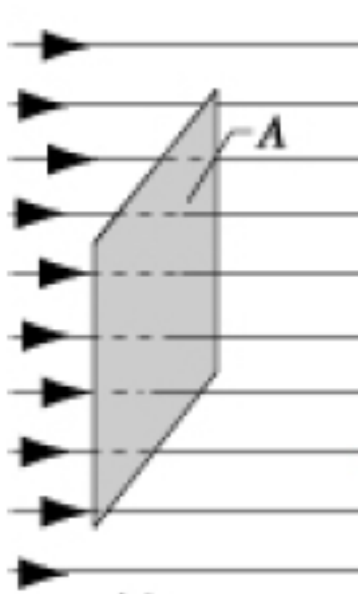
IMPORTANT: LABS START NEXT WEEK

- Gauss' law
  - The flux of a vector field
  - Electric flux and field lines
- Gauss' law for a point charge
- The shell theorem
  - Examples involving spherical charge distributions
- Gauss' law for other symmetries
  - A uniformly charged sheet
- Gauss' law and conductors

Reading: up to page 363 in the text book (end Ch. 21)

Flux of a vector field (Ch. 21)





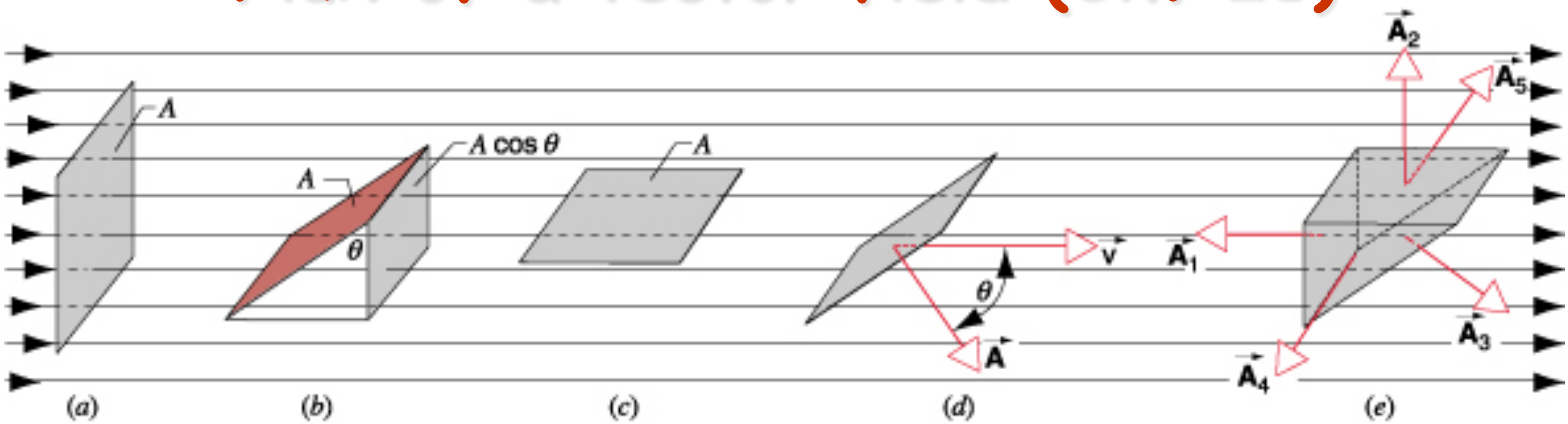
Consider the velocity field of a flowing fluid

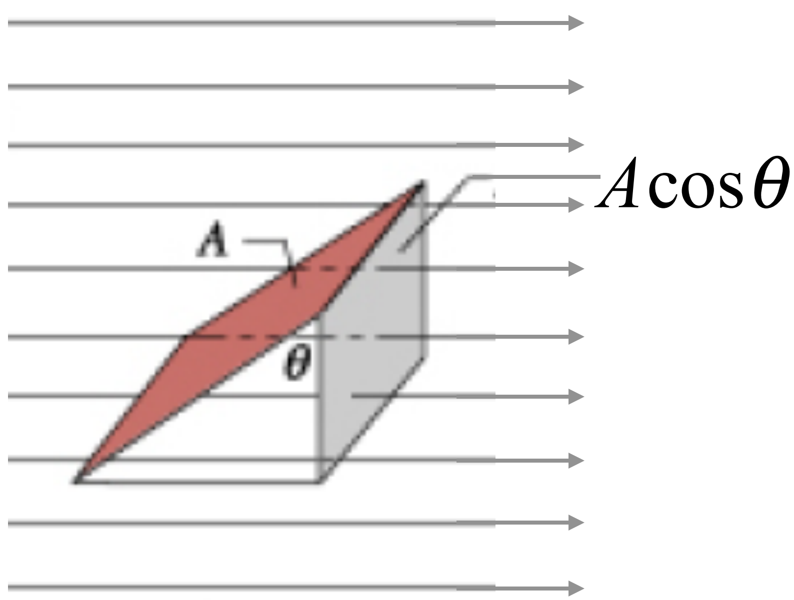
If the area A is flat, and perpendicular to the flow, then we define the flux $\mathbf{\Phi}$ of the velocity field $\vec{\mathbf{v}}$ through the surface as follows:

$$|\Phi| = vA$$

If v has dimensions of m/s, then the flux has dimensions of m^3/s (**volume flow rate**).

Flux of a vector field (Ch. 21)



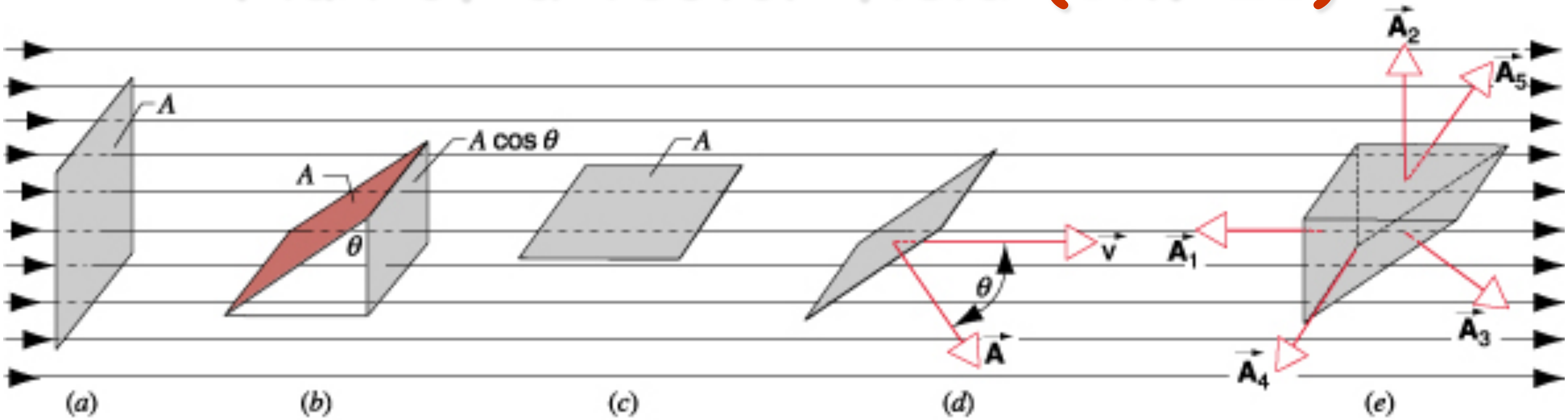


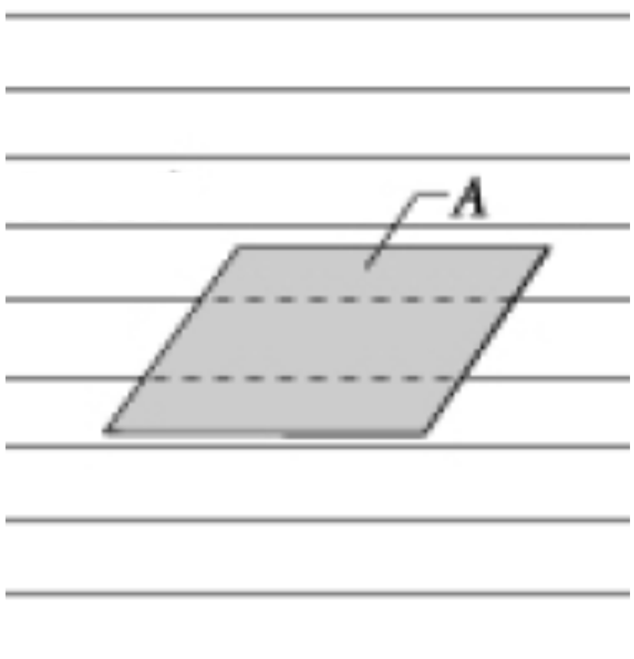
What if the area is not perpendicular to the flow?

We project the area on to the plane perpendicular to the flow. Then,

$$|\Phi| = vA\cos\theta$$

Flux of a vector field (Ch. 21)



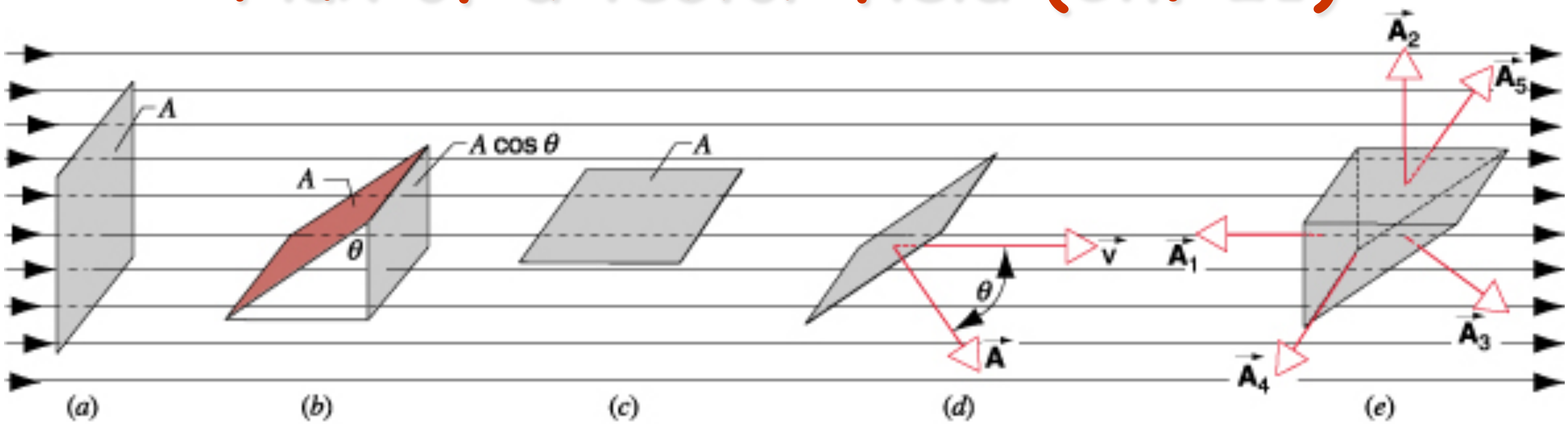


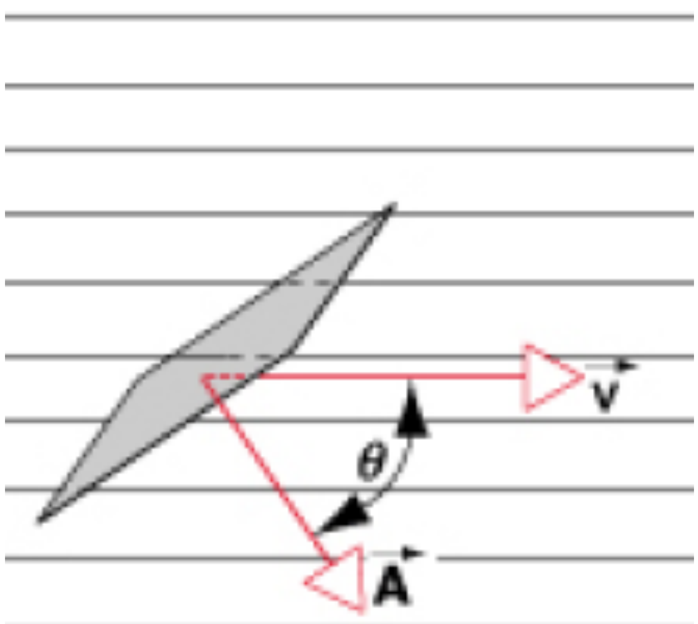
What if the area is not perpendicular to the flow?

For this extreme case ($\theta = 90^o$),

$$|\Phi| = 0.$$

Flux of a vector field (Ch. 21)





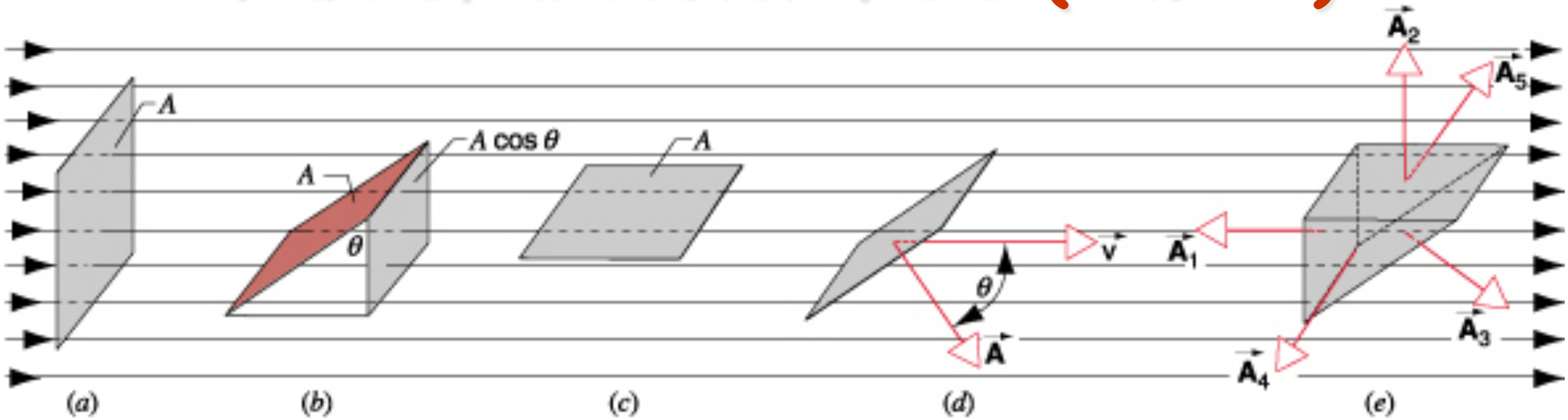
What if the area is not perpendicular to the flow?

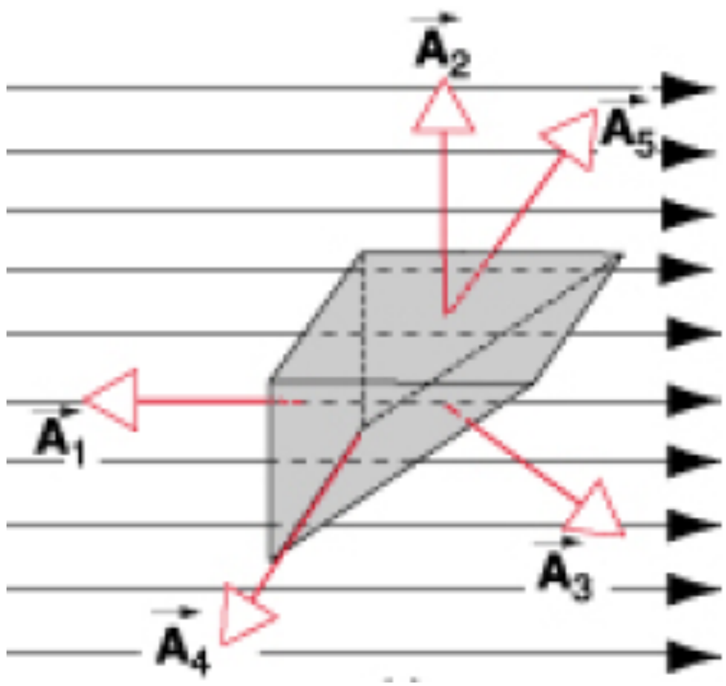
In general,

$$\left|\Phi\right| = \vec{\mathbf{v}} \cdot \vec{\mathbf{A}},$$

where $\vec{\mathbf{A}}$ has the magnitude of A and is oriented perpendicular to the surface.

Flux of a vector field (Ch. 21)





What if there are multiple surface elements to consider?

Then,

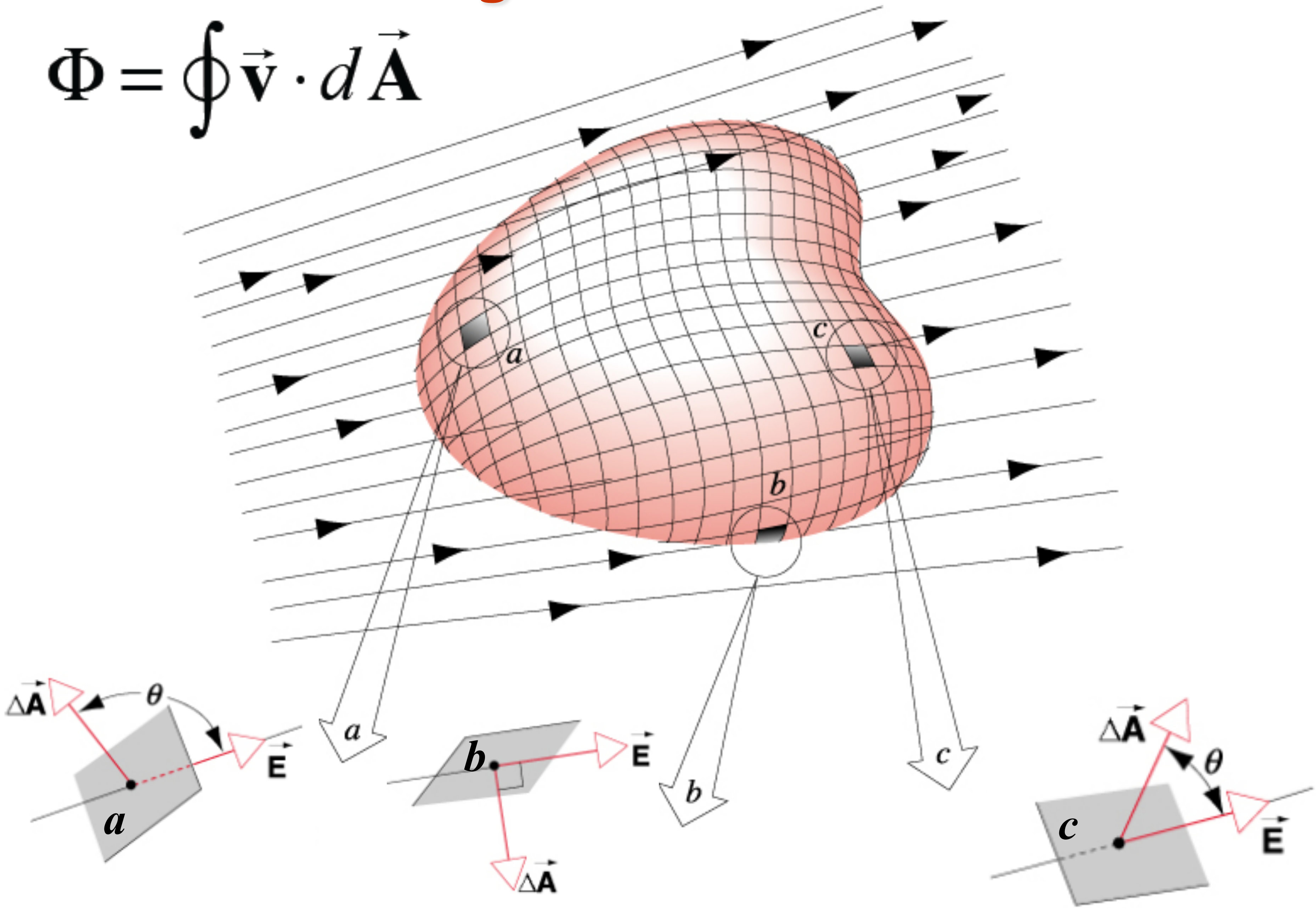
sign now determined

$$\Phi = \sum \vec{v} \cdot \vec{A}.$$

For a closed surface, we ALWAYS choose $\vec{A}$ to point outwards. This is very important for Gauss' Law!!

The flux through a closed curved surface

$$\Phi = \oint \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}}$$



The flux of an electric field

Gauss' law is concerned with the flux of E through closed surfaces

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

- **You may recall that when we developed our graphical representation of electric field lines, the electric field strength was proportional to the number of field lines crossing a unit area perpendicular to the field.**
- **Consequently,**

$$Flux = \sum \frac{\#of\ \ field\ lines}{\perp area} \times (\perp area) = \sum \#of\ \ field\ lines.$$

- **In other words, the flux of E through a surface is proportional to the number of field lines penetrating the surface.**
- **This is the essence of Gauss' law.**
- **Recall also that the number of field lines is related to the number of charges producing the electric field.**

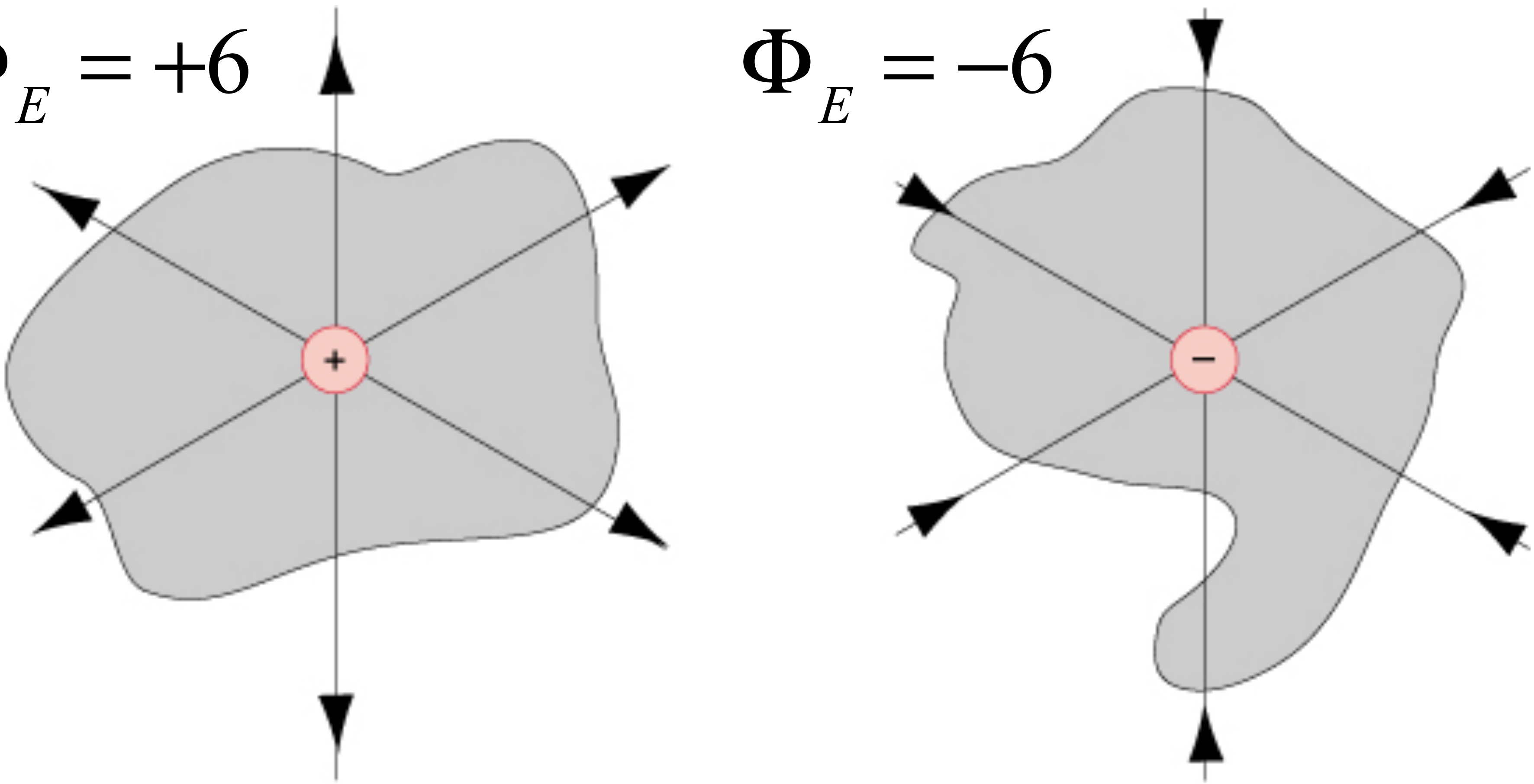
The flux of an electric field

Gauss' law is concerned with the flux of E through closed surfaces

$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}}$$

$$\Phi_E = +6$$

$$\Phi_E = -6$$



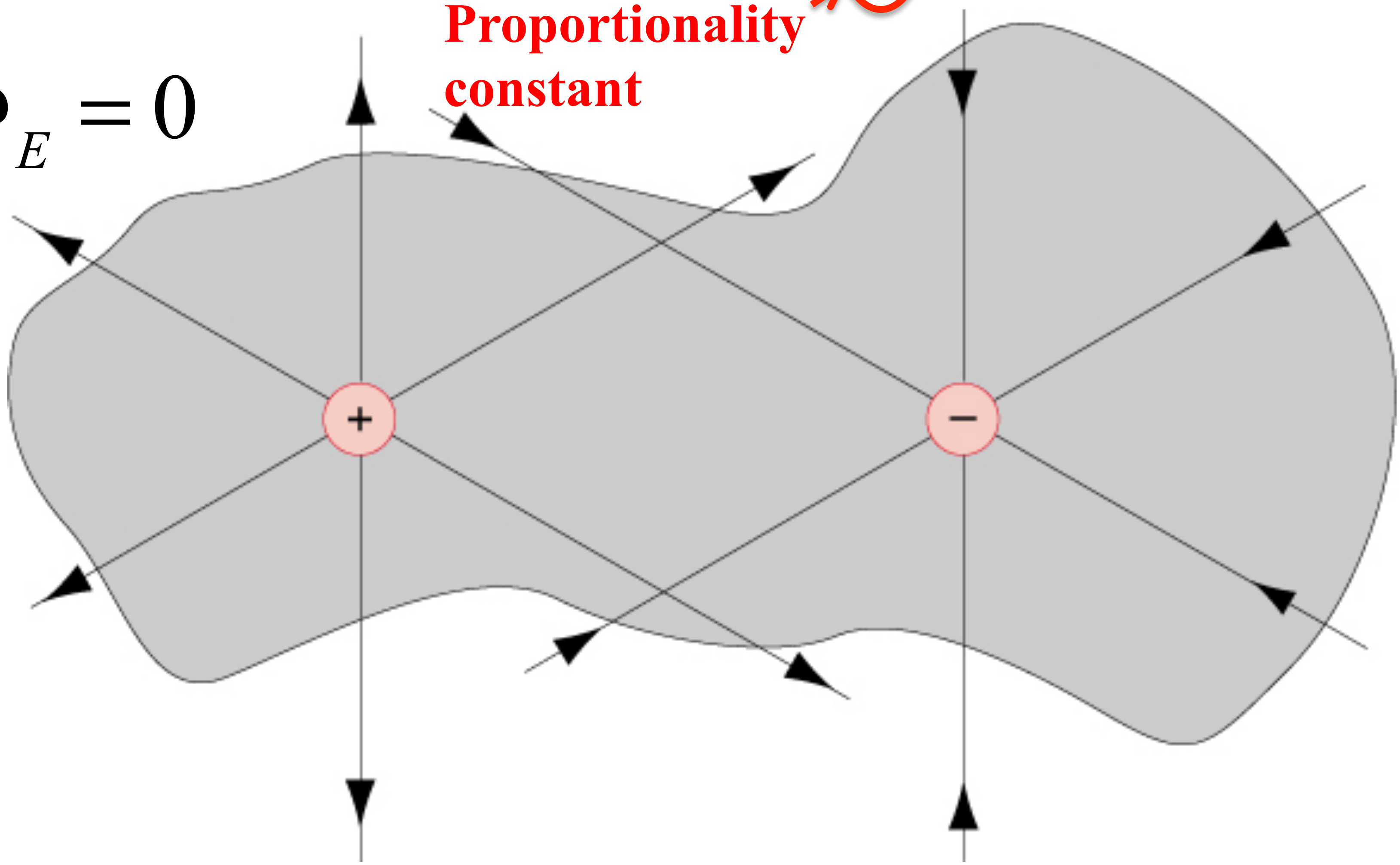
The flux of an electric field

Gauss' law is concerned with the flux of E through closed surfaces

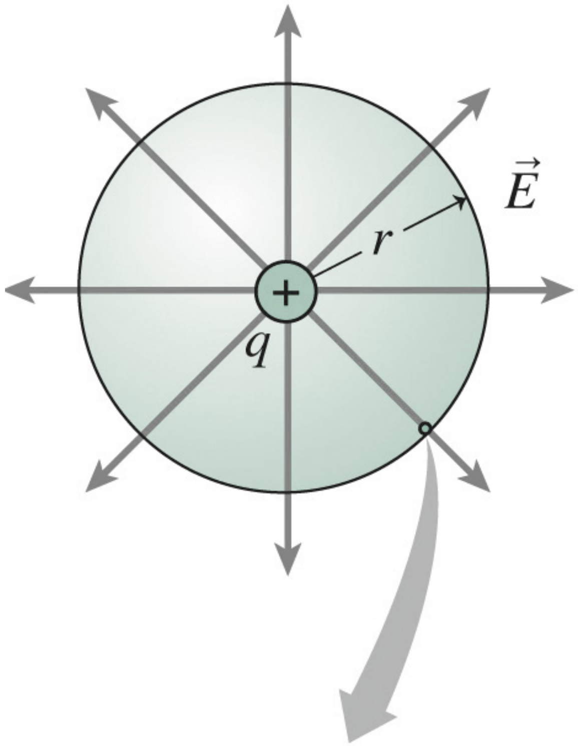
$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{1}{\varepsilon_o} \sum q_{enclosed}$$

Proportionality constant

$$\Phi_E = 0$$



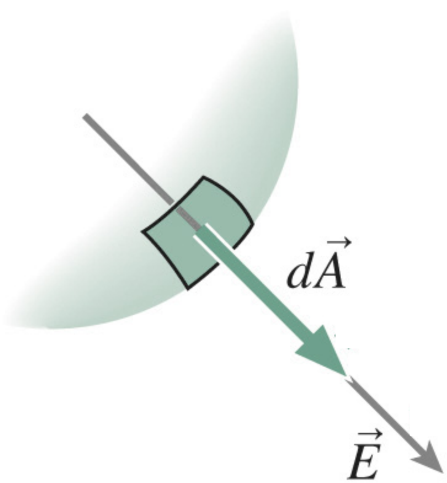
Gauss' law for a point charge



$$\Phi_E = \oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_o}$$

Exploit symmetry:

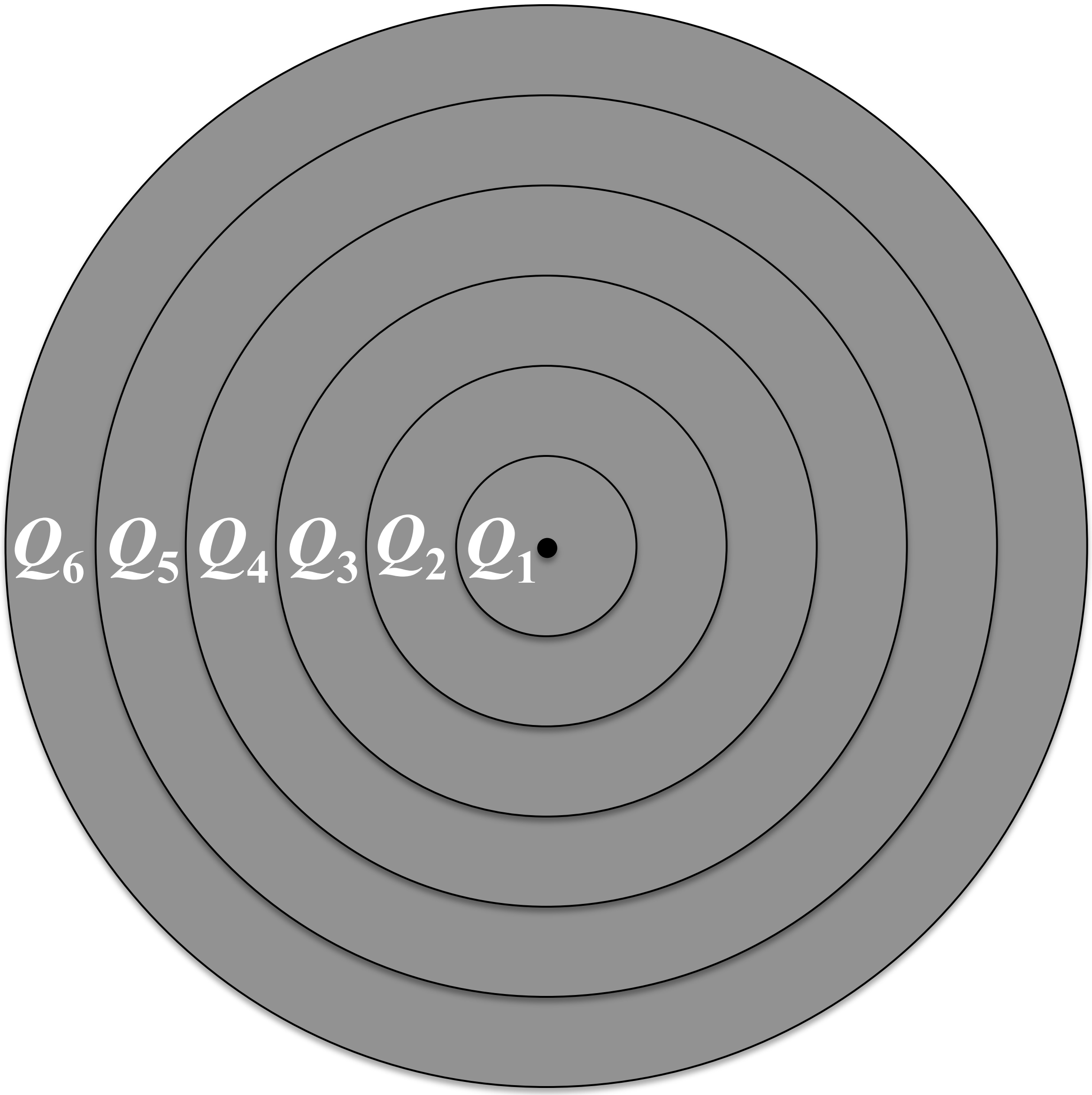
- Invent an imaginary spherical surface.
- Electric field will be perpendicular to all points on this surface.
- Electric field will have the same strength at all points on surface
- Greatly simplifies integration



$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \oint E\, dA = E \oint dA = E \times 4\pi r^2$$

$$\Rightarrow E \times 4\pi r^2 = \frac{q}{\varepsilon_o} \quad \text{or} \quad \vec{\boldsymbol{E}} = \frac{q}{4\pi\varepsilon_o r^2}\hat{\boldsymbol{r}} = k\frac{q}{r^2}\hat{\boldsymbol{r}}$$

Gauss' law and the shell theorem

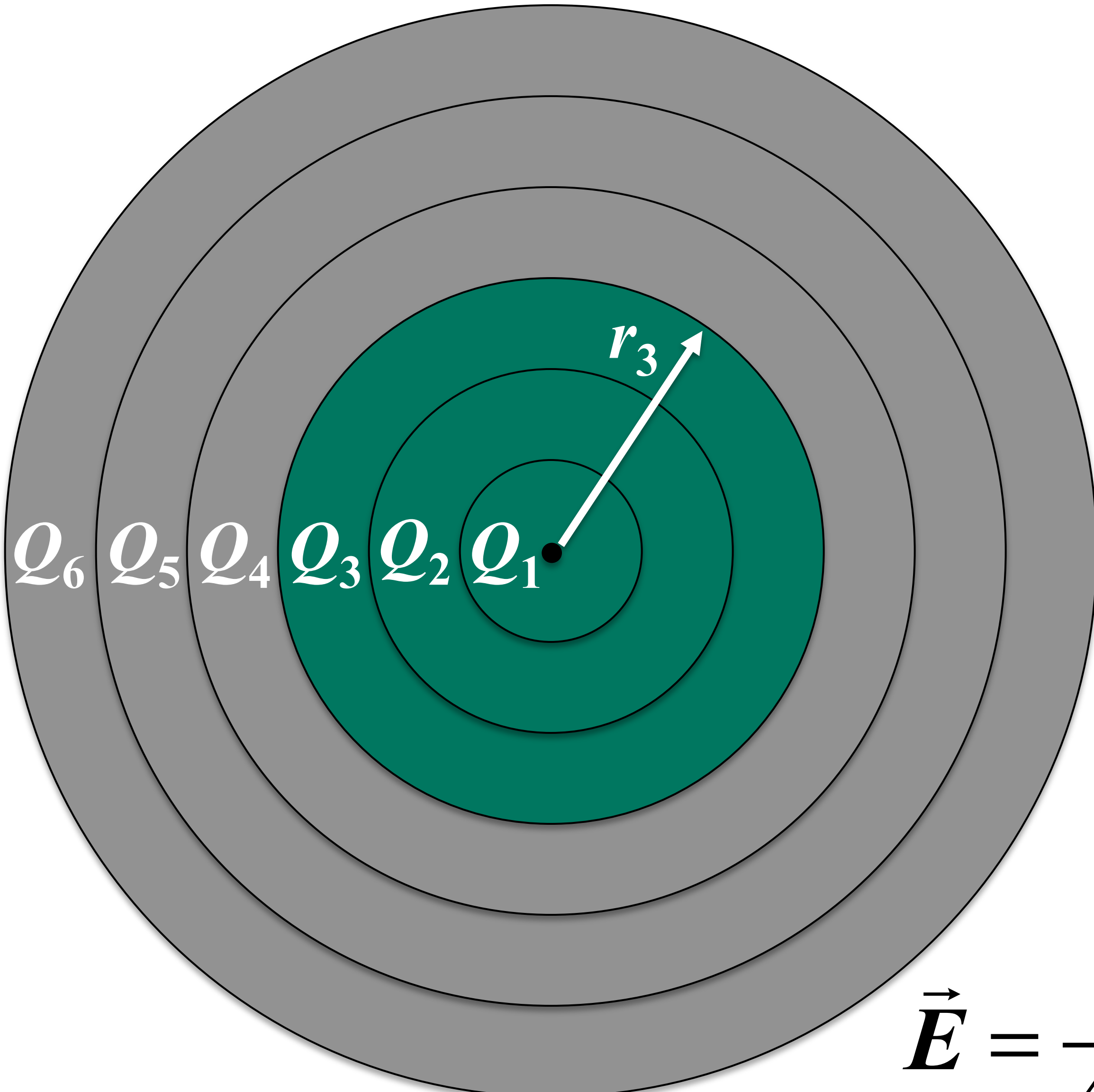


$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_o}$$

$$4\pi r^2 E = \frac{q_{enc}}{\varepsilon_o}$$

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_o}\frac{q_{enc}}{r^2}\hat{\boldsymbol{r}}$$

Gauss' law and the shell theorem



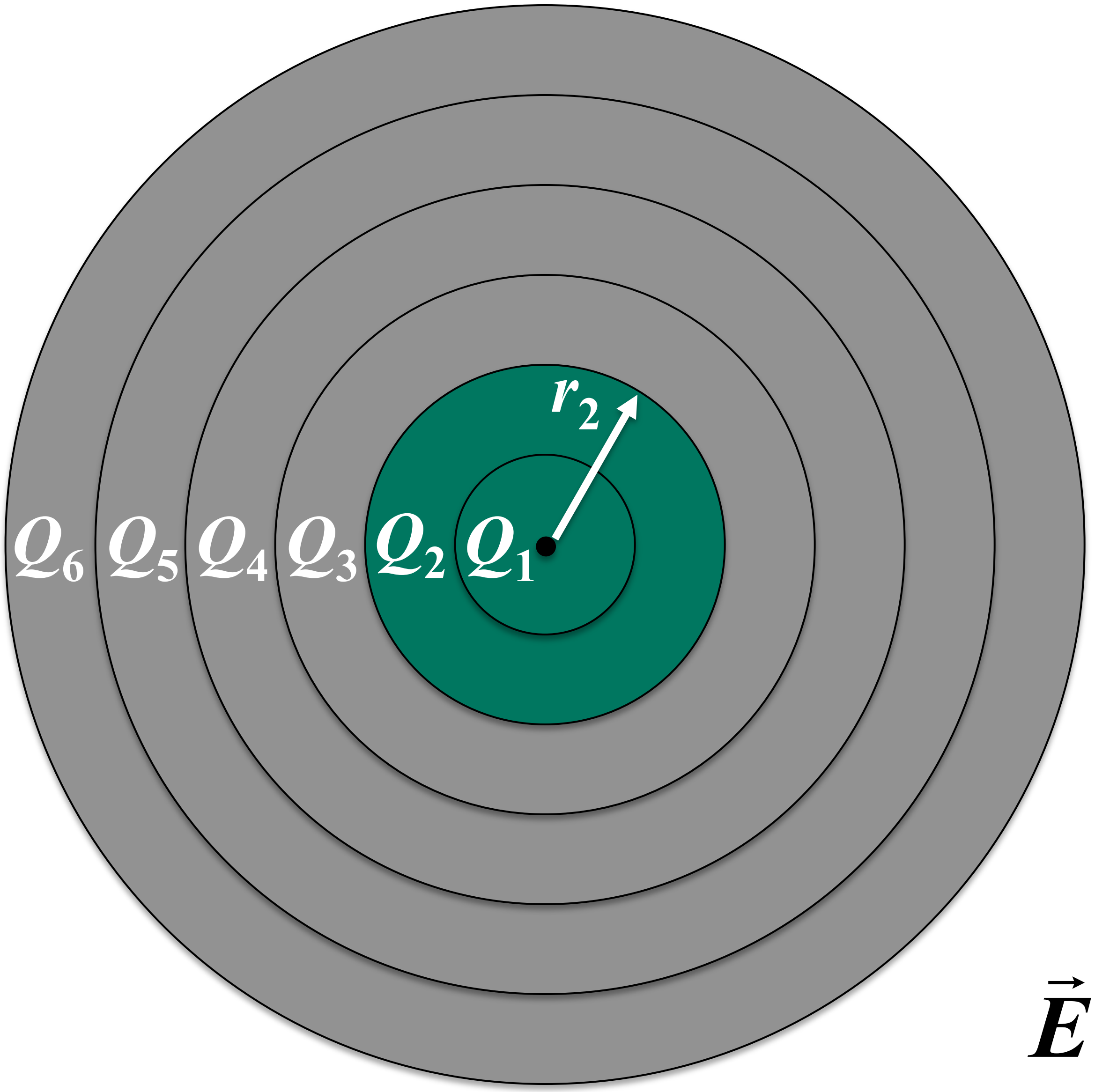
$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_o}$$

$$4\pi r^2 E = \frac{q_{enc}}{\varepsilon_o}$$

$$q_{enc} = Q_1 + Q_2 + Q_3$$

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_o} \frac{Q_1 + Q_2 + Q_3}{r_3^2} \hat{\boldsymbol{r}}$$

Gauss' law and the shell theorem



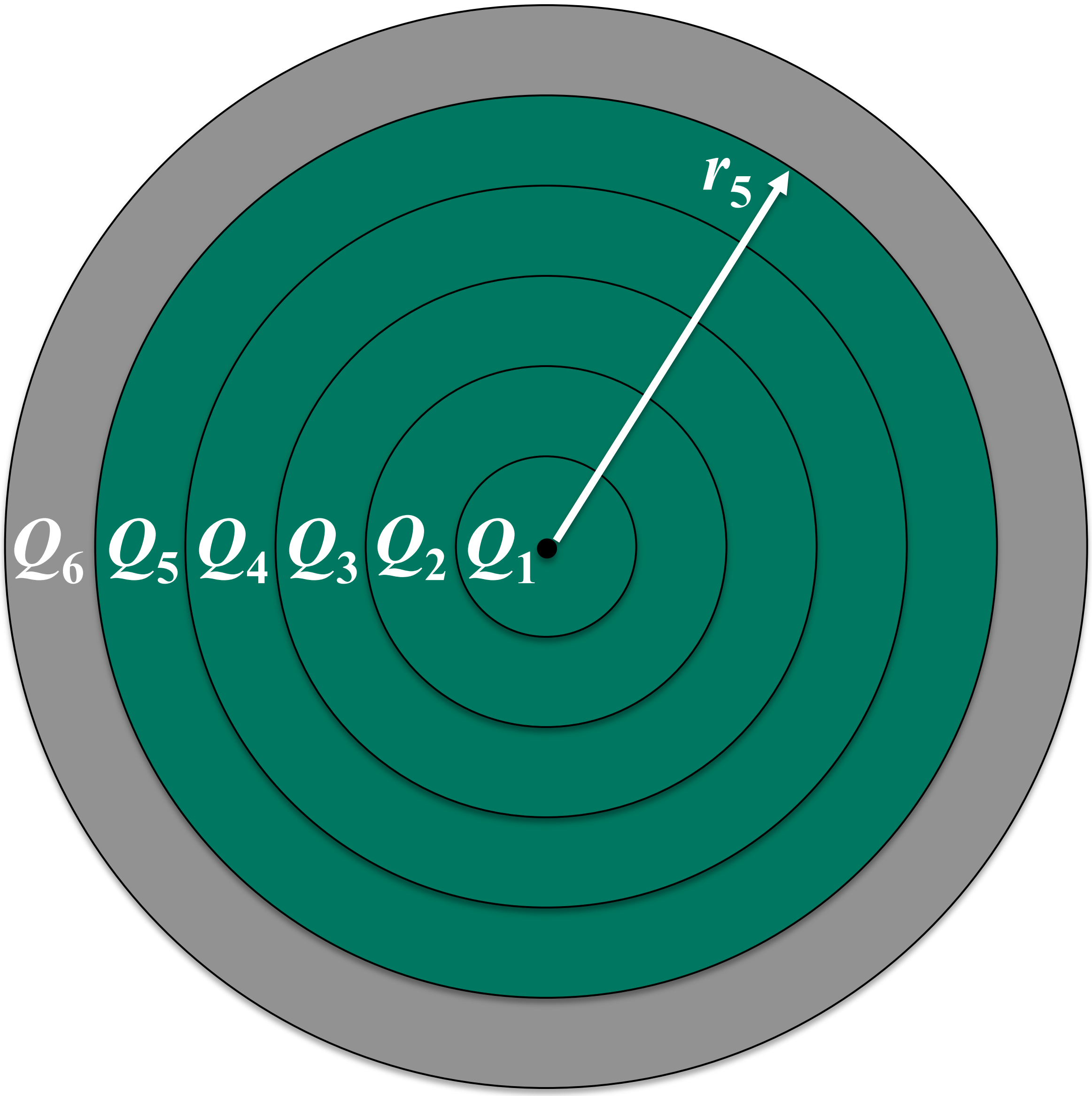
$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_o}$$

$$4\pi r^2 E = \frac{q_{enc}}{\varepsilon_o}$$

$$q_{enc} = Q_1 + Q_2$$

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_o}\frac{Q_1 + Q_2}{r_2^2}\hat{\boldsymbol{r}}$$

Gauss' law and the shell theorem



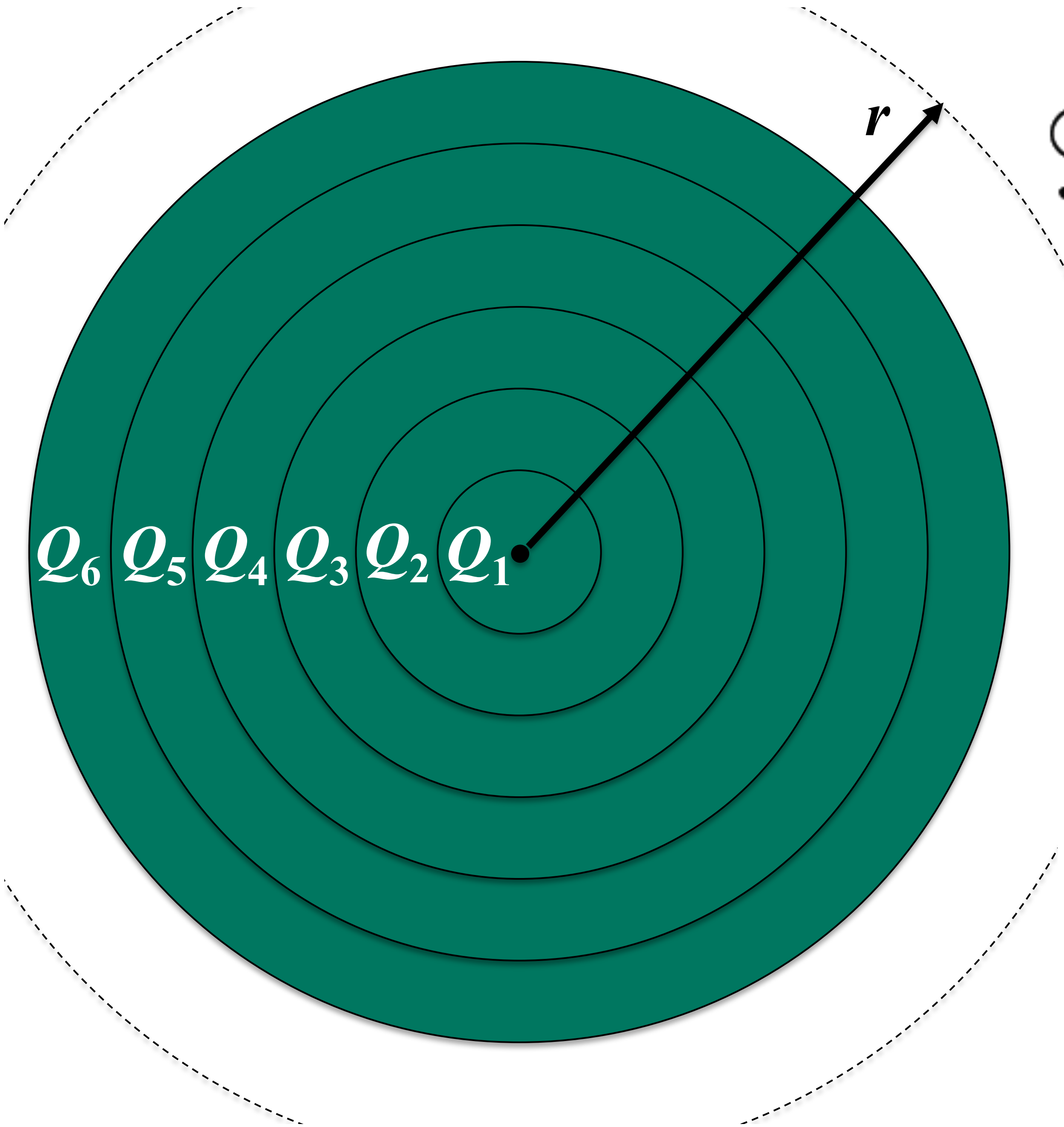
$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_o}$$

$$4\pi r^2 E = \frac{q_{enc}}{\varepsilon_o}$$

$$q_{enc} = \sum_{i=1}^{5} Q_i$$

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_o} \frac{\sum_{i=1}^{5} Q_i}{r_5^2} \hat{\boldsymbol{r}}$$

Gauss' law and the shell theorem



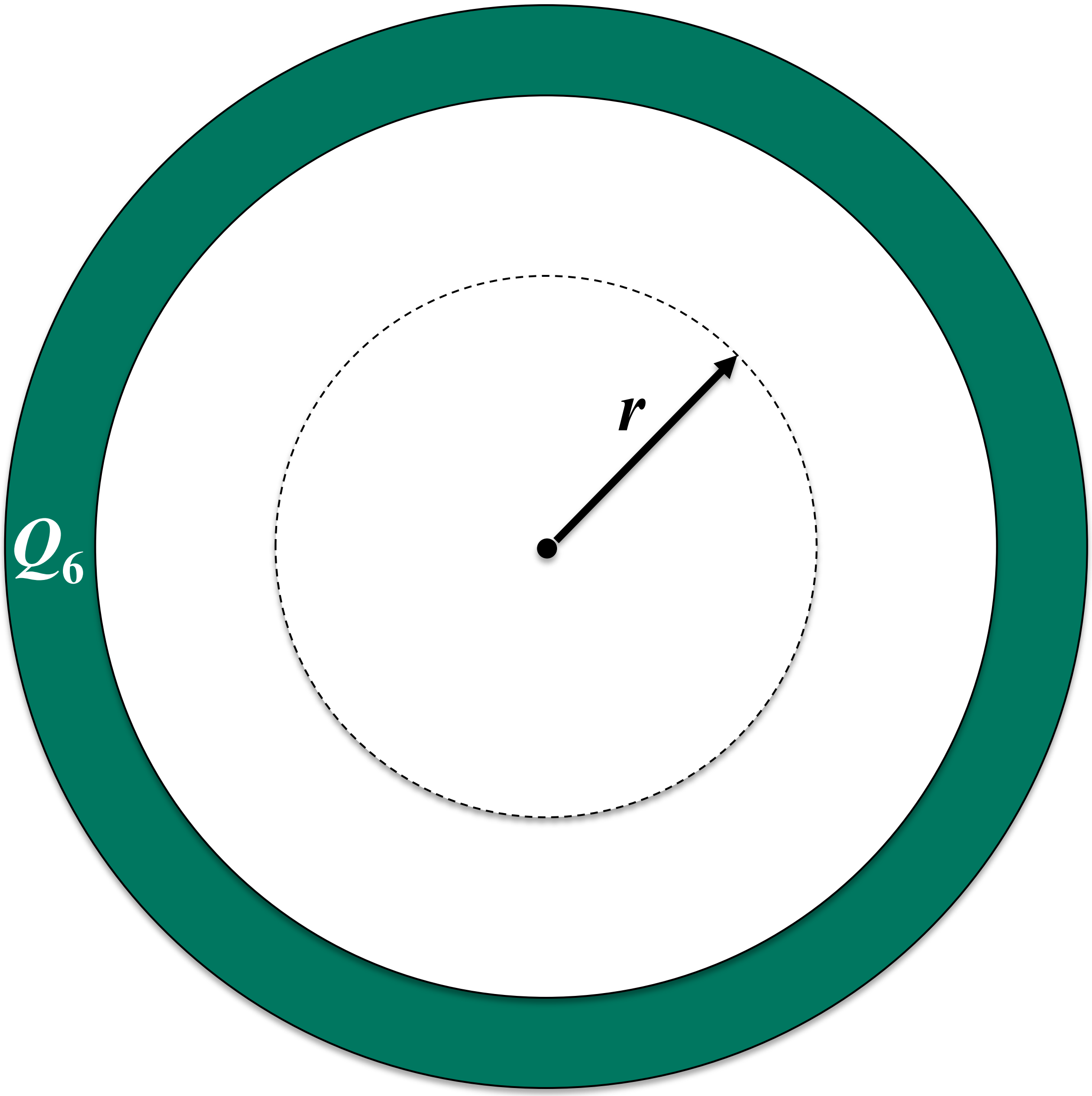
$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_o}$$

$$4\pi r^2 E = \frac{q_{enc}}{\varepsilon_o}$$

$$q_{enc} = \sum_{i=1}^{6} Q_i = Q_{tot}$$

$$\vec{\boldsymbol{E}} = \frac{1}{4\pi\varepsilon_o}\frac{Q_{tot}}{r^2}\hat{\boldsymbol{r}}$$

Gauss' law and the shell theorem



$$\oint \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{enc}}{\varepsilon_o}$$

$$4\pi r^2 E = \frac{q_{enc}}{\varepsilon_o}$$

$$q_{enc} = 0$$

$$\vec{E} = 0$$

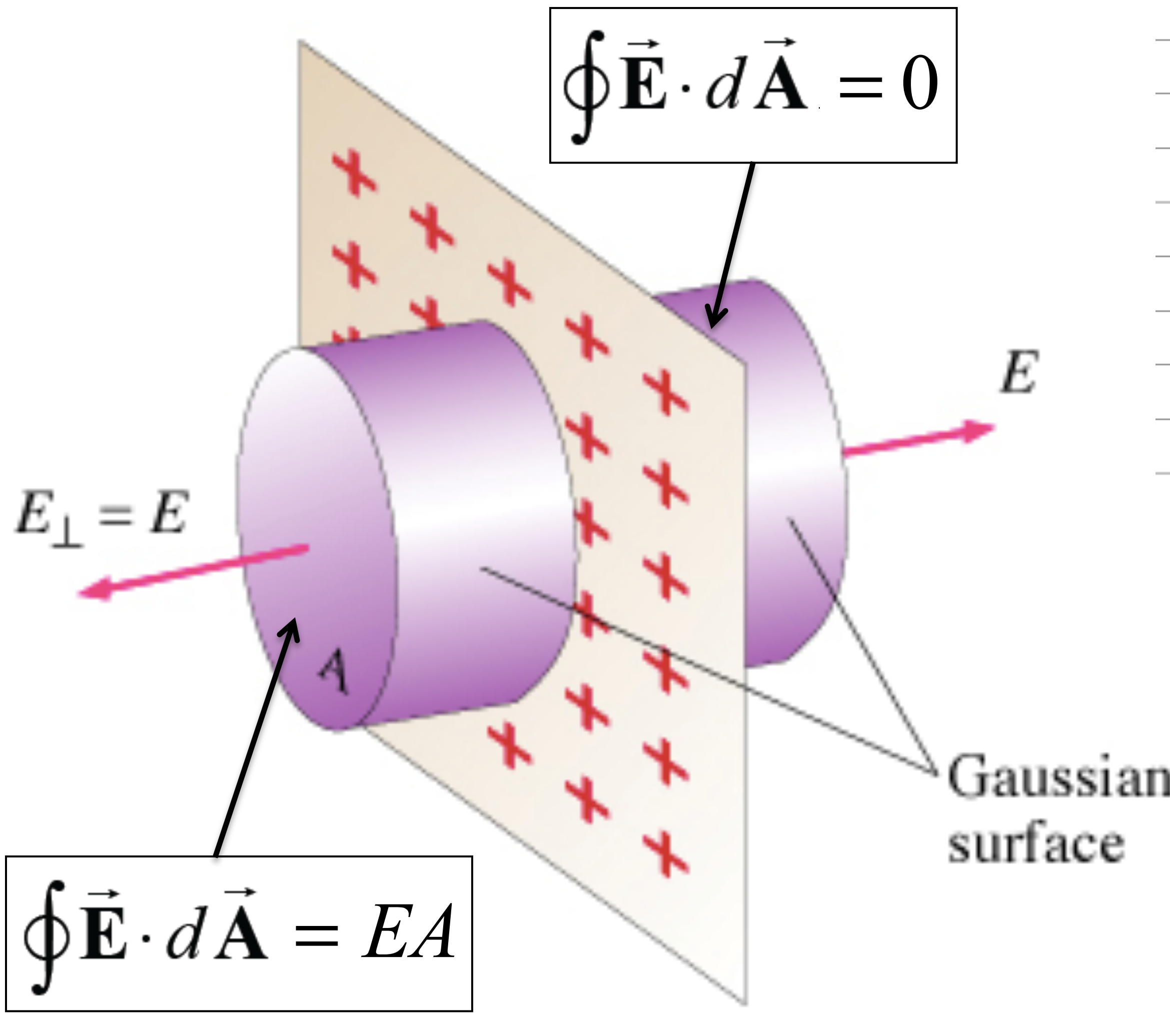
Gauss' law and the shell theorem

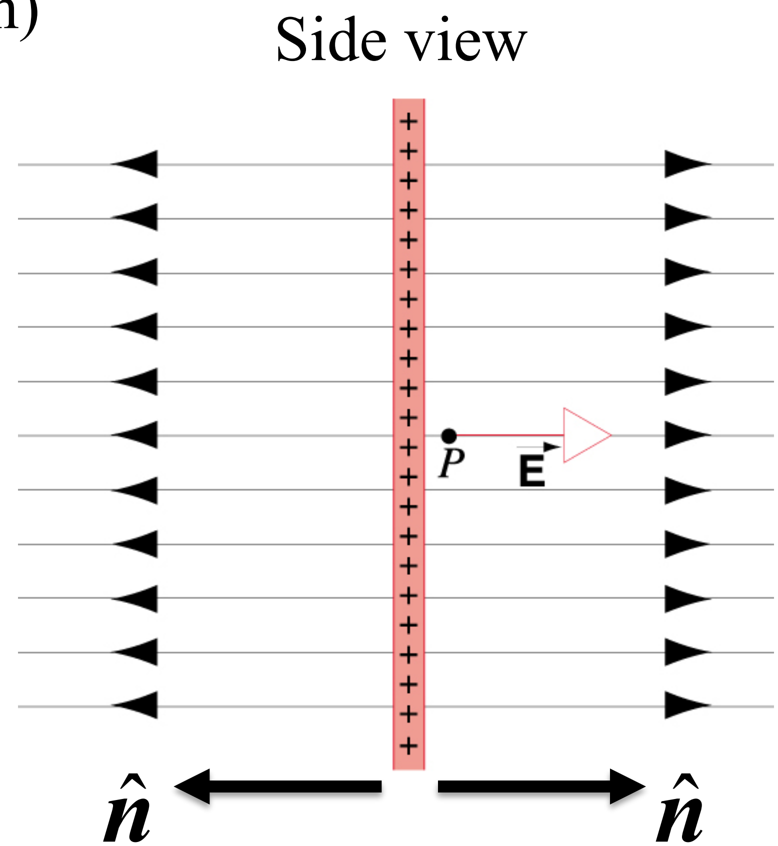
Key ideas:

- **Symmetry is crucial – symmetries that work:**
  - **Spherical (solid sphere, spherical shell, etc..)**
  - **Cylindrical (line charge, tube of charge, etc..)** **Next week**
  - **Planar (sheet of charge, slab of charge, etc..)**
- **Construct an imaginary (Gaussian) surface to aid in calculating the field; you then calculate the flux through this surface.**
- **For the spherical case, the Gaussian surface must be spherical and concentric with the charge, otherwise the surface integral is undetermined (similar principles apply to the other symmetries).**
- **The flux through the Gaussian surface and, therefore, the field, depends only on the charge inside the surface;**
  - **The electric field at radius $r = R$ knows nothing about charge at larger radii, $r > R$;**
  - **If all charge is contained within a radius R then, for radii $r > R$, it appears as though all of the charge is located at the center of the sphere, i.e., field knows nothing about charge distribution.**

Gauss' law for sheet of charge

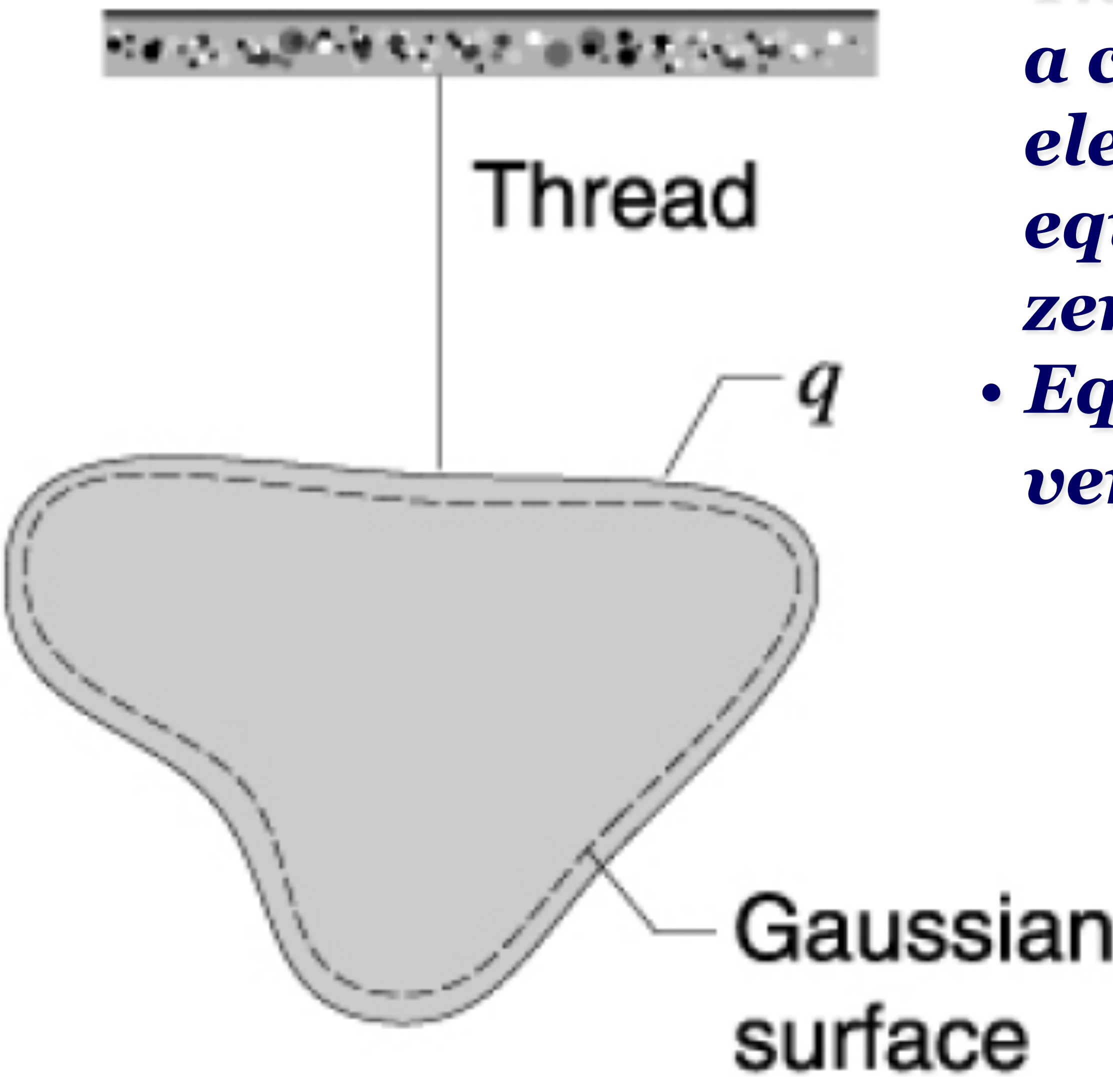
- Uniform field (does not depend on position)
- Everywhere perpendicular to the surface
- Surface charge density, σ; units are C/m^2





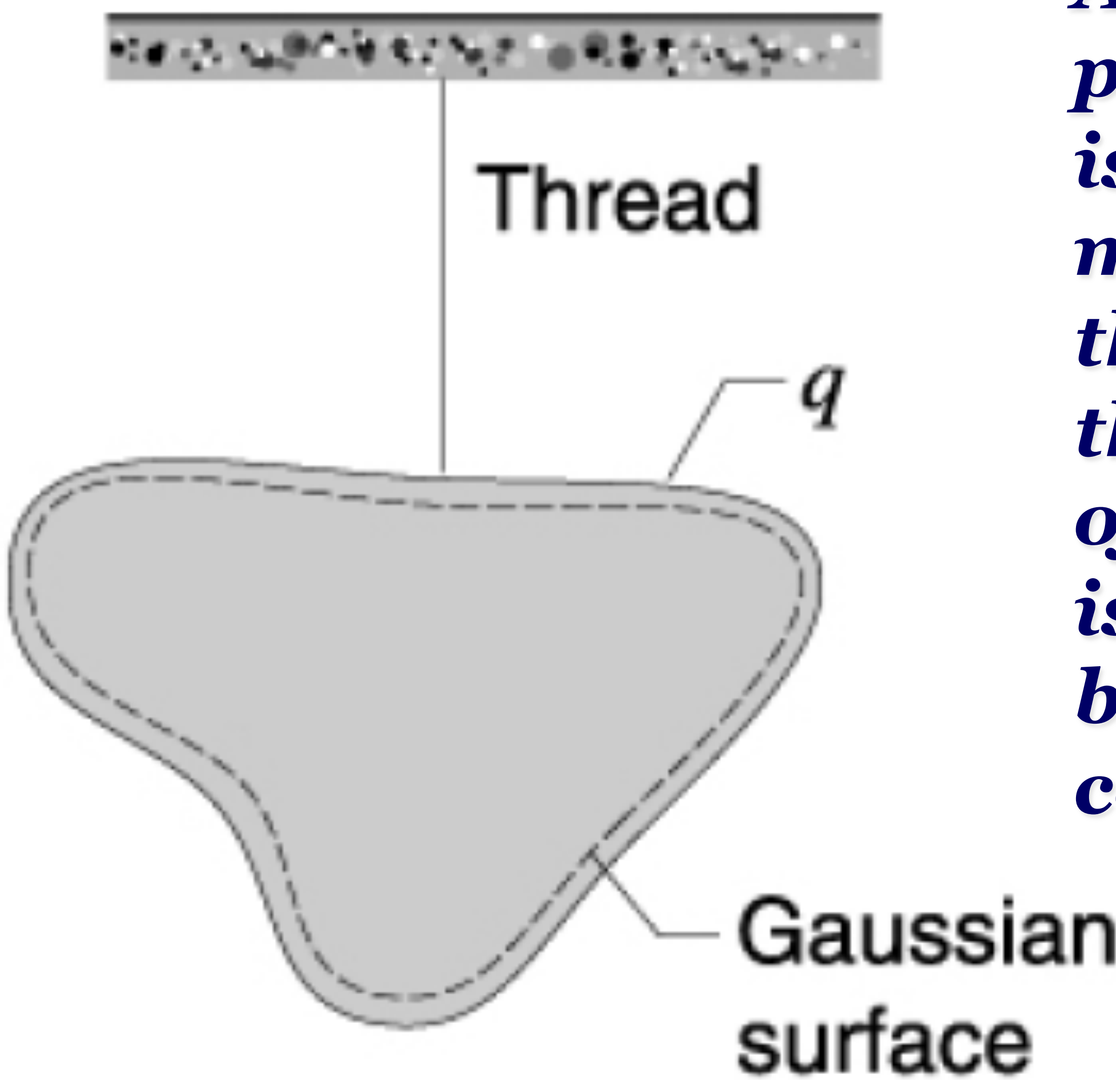
$$\vec{E} = \frac{\sigma}{2\varepsilon_o}\hat{n}$$

Gauss' law and conductors



- ***The electric field inside a conductor which is in electrostatic equilibrium must be zero.***
- ***Equilibrium is reached very quickly (<10^{-9} s).***

Gauss' law and conductors



An excess charge placed on an isolated conductor moves entirely to the outer surface of the conductor. None of the excess charge is found within the body of the conductor.

Charge densities

In 1D (a line or wire):

$$\lambda = \frac{Q}{L}, \qquad \text{or} \qquad \lambda = \frac{dQ}{dL}$$

λ is the line charge density, or charge per unit length, in Coulombs per meter. L represents length, and Q is charge.

In 2D (a surface or sheet):

$$\sigma = \frac{Q}{A}, \qquad \text{or} \qquad \sigma = \frac{dQ}{dA}$$

σ is the surface charge density, or charge per unit area in Coulombs per meter^2; A represents area, and Q is charge.

In 3D (a solid object):

$$\rho = \frac{Q}{V}, \qquad \text{or} \qquad \rho = \frac{dQ}{dV}$$

ρ is the volume charge density, or charge per unit volume in Coulombs per meter^3. V represents volume, and Q is charge.